

INVERSE SCATTERING FOR THE NONLINEAR SCHRÖDINGER EQUATION WITH THE YUKAWA POTENTIAL

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ABSTRACT. We study the inverse scattering problem for the three dimensional nonlinear Schrödinger equation with the Yukawa potential. The nonlinearity of the equation is nonlocal. We reconstruct the potential and the nonlinearity by the knowledge of the scattering states. Our result is applicable to reconstructing the nonlinearity of the semi-relativistic Hartree equation.

1. INTRODUCTION

We consider the inverse scattering problem for the three dimensional nonlinear Schrödinger equation

$$i\partial_t u + \Delta u + Q_0 \frac{\exp(-\mu_0 r)}{r} u - \left(Q_1 \frac{\exp(-\mu_1 r)}{r} * |u|^2 \right) u = 0 \quad (\text{NLS})$$

in $\mathbb{R} \times \mathbb{R}^3$. Here, u is a complex-valued unknown function of $(t, x) \in \mathbb{R} \times \mathbb{R}^3$, $\partial_t = \partial/\partial t$, Δ is the Laplacian in \mathbb{R}^3 , $r = |x|$, $Q_0, Q_1 \in \mathbb{R}^3$, $\mu_0, \mu_1 > 0$ and $*$ is the convolution in the space variables. Recall that the functions

$$V_j := -Q_j \frac{\exp(-\mu_j r)}{r}, \quad j = 0, 1,$$

are said to be the Yukawa potential. The equation (NLS) is approximately derived from the generalization of the electronic Hamiltonian for an N -electron atom in a plasma:

$$\mathbf{H} = -\frac{1}{2} \sum_{j=1}^N \Delta_j - \sum_{j=1}^N \frac{Z \exp(-\mu_0 |x^j|)}{|x^j|} + \sum_{j>k}^N \frac{\exp(-\mu_1 |x^j - x^k|)}{|x^j - x^k|},$$

where $x^j \in \mathbb{R}^3$ is the space variables for the j -th particle, Δ_j is the Laplacian with respect to x^j , Z is the nuclear charge and μ_l , $l = 0, 1$, are parameters depending on the density and the temperature of the plasma (see, e.g., Mukherjee–Karwowski–Diercksen [11]).

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In order to mention the inverse scattering problem, we introduce the definition of the scattering operator for the nonlinear evolution equation

$$i\partial_t v(t) + J(v(t)) = f(v(t)), \quad t \in \mathbb{R}, \quad (1.1)$$

where v is a complex-valued function on the Hilbert space X , J is a self-adjoint operator on X and f is a perturbed term. Let $B(\delta; X)$ be the set of all $\phi \in X$ with $\|\phi\|_X \leq \delta$. The scattering operator S is defined by the mapping

$$S : B(\delta; X) \ni \phi_- \mapsto \phi_+ \in X$$

if the following condition holds for some $\delta > 0$ and some $Z \subset C(\mathbf{R}; X)$:

For any $\phi_- \in B(\delta; X)$, there uniquely exists $v \in Z$ such that v is a time-global solution to (1.1) and satisfies

$$\lim_{t \rightarrow -\infty} \|v(t) - e^{itJ}\phi_-\|_X = 0.$$

Furthermore, there uniquely exists $\phi_+ \in X$ such that

$$\lim_{t \rightarrow \infty} \|v(t) - e^{itJ}\phi_+\|_X = 0.$$

We remark that $e^{itJ}\phi$ is a solution to the Cauchy problem for

$$\begin{cases} i\partial_t v(t) + J(v(t)) = 0, & t \in \mathbb{R}, \\ v(0) = \phi. \end{cases}$$

The inverse scattering problem for the equation (1.1) is to recover the perturbed term f by applying the knowledge of the scattering operator S . Before we treat (NLS), we first review the inverse scattering problem for the Schrödinger equation with power nonlinearity briefly. Strauss [19] considered the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = V(x)|u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.$$

Suppose that p is an integer satisfying

$$\begin{cases} p > 4 & \text{if } n = 1, \\ p > 3 & \text{if } n = 2, \\ p \geq 3 & \text{if } n \geq 3, \end{cases}$$

and $V(x)$ is real-valued continuous and bounded, whose derivatives up to order $l > 3n/4$ are bounded. Then the scattering operator S is well-defined. It was shown that $V(x)$ is recovered from the scattering operator by the following way: For $s \in \mathbb{R}^n$, let $H^s(\mathbb{R}^n)$ be the Sobolev space $(1 - \Delta)^{-s/2}L^2(\mathbb{R}^n)$. For any $\phi \in H^1(\mathbb{R}^n) \cap L^{1+1/p}(\mathbb{R}^n)$, we have

$$V(x_0) = \frac{\lim_{\alpha \rightarrow 0} \alpha^{-(n+2)} I[\phi_{\alpha, x_0}]}{\int_{\mathbb{R}} \int_{\mathbb{R}^n} |e^{it\Delta}\phi(x)|^{p+1} dx dt}, \quad (1.2)$$

where $\phi_{\alpha, x_0}(x) = \phi(\alpha^{-1}(x - x_0))$, $\alpha > 0$, $x, x_0 \in \mathbb{R}^n$ and

$$I[\phi] = \lim_{\varepsilon \rightarrow 0} \frac{i}{\varepsilon^p} \langle (S - id)(\varepsilon\phi), \phi \rangle_{L^2(\mathbb{R}^n)}.$$

The above limit is called the small amplitude limit. Later, Weder [25, 27, 28, 29, 31] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [19].

Unfortunately, the above methods to obtain the reconstruction formulas are not applicable to the case (NLS) even if $Q_0 = 0$. The essential point to prove the formula (1.2) is the change of variables in the following integral:

$$I[\phi] = \int_{\mathbb{R}} \int_{\mathbb{R}^n} V(x) |e^{it\Delta} \phi(x)|^{p+1} dx dt.$$

By changing variable x by $\alpha^{-1}(x - x_0)$, we have

$$I[\phi_{\alpha, x_0}] = \alpha^{n+2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} V(x_0 + \alpha x) |e^{it\Delta} \phi(x)|^{p+1} dx dt.$$

Therefore, as $\alpha \rightarrow 0$, we can take the value $V(x_0)$ from the inside integral. Applying the same method to (NLS) with $Q_0 = 0$, we obtain

$$\begin{aligned} I[\phi_{\alpha, x_0}] &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} (V * |e^{it\Delta} \phi_{\alpha, x_0}|^2)(x) |e^{it\Delta} \phi_{\alpha, x_0}(x)|^2 dx dt \\ &= \alpha^{2n+2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} (V(\alpha \cdot) * |e^{it\Delta} \phi|^2)(x) |e^{it\Delta} \phi(x)|^2 dx dt, \end{aligned}$$

where $V(x) = Q_1 \frac{\exp(-\mu_1 r)}{r}$. Since the integral

$$\int_{\mathbb{R}} \int_{\mathbb{R}^n} (V(0) * |e^{it\Delta} \phi|^2)(x) |e^{it\Delta} \phi(x)|^2 dx dt$$

does not converge, we can not make α tend to infinity.

We next review the inverse scattering problem for the nonlinear Schrödinger equation with a cubic convolution

$$i\partial_t u + \Delta u + \tilde{V}(x)u = F^\sigma(u), \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n. \quad (1.3)$$

Here, $\tilde{V} : \mathbb{R}^3 \rightarrow \mathbb{C}$ is measurable and satisfies some suitable condition,

$$F^\sigma(u) = \lambda(x)(|\cdot|^{-\sigma} * |u|^2)u$$

and $\lambda \in C^1(\mathbf{R}^n) \cap W_\infty^1(\mathbf{R}^n)$. It was proved by Watanabe [21] that if σ is a given number, then we can reconstruct V and λ by the knowledge of the scattering operator. Watanabe [23] determined σ of the term F^σ if $\tilde{V} \equiv 0$ and $\lambda(x)$ is a non-zero constant

function. Under the condition $\tilde{V} \equiv 0$, Sasaki [17] proved that σ of F^σ can be determined even if λ_j is not a constant. In fact, σ is given by

$$\sigma = 2n + 2 - \lim_{\alpha \rightarrow 0} \ln \frac{|T[\phi_{e\alpha}]|}{|T[\phi_\alpha]| + \alpha^{2n+2}}, \quad (1.4)$$

$$T[\phi] = \lim_{\varepsilon \rightarrow 0} \frac{i}{\varepsilon^3} \langle (S - id)(\varepsilon\phi), \phi \rangle_{L^2(\mathbb{R}^n)},$$

where e is the base of the natural logarithm, $\phi \in H^1(\mathbb{R}^n) \setminus \{0\}$, $\phi_\alpha = \phi_{\alpha,0}$ and S is the scattering operator. For other results of the inverse scattering problem for (1.3), see Watanabe [22, 24] and Sasaki–Watanabe [18].

As we mention before, we study the inverse scattering problem for (NLS). Remark that we can not directly apply the known results to recovering the functions V_j , $j = 0, 1$. Our goal in this paper is to give a formula for determining the parameter Q_j and μ_j , $j = 0, 1$, by using the knowledge of the scattering operator for (NLS) given by Theorem 1.1 below.

We now define some notation which will be used later. Let $\mathbb{Z}_{\geq 0}$ be the set of all non-negative integers. For $\Omega \subset \mathbb{R}^n$, let $C_c^\infty(\Omega)$ be the set of all smooth functions with compact support in Ω . We put $L^2(\mathbb{R}^3) = \mathcal{H}$. We denote the norm and the inner product of \mathcal{H} by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$, respectively. For $1 \leq p, q \leq \infty$, $\|\cdot\|_q$ and $\|\cdot\|_{(p,q)}$ denote $\|\cdot\|_{L^q(\mathbb{R}^3)}$ and $\|\cdot\|_{L^p(\mathbb{R}; L^q(\mathbb{R}^3))}$, respectively. We set $F(u) = -(V_1 * |u|^2)u$. Let H be an unbounded operator on \mathcal{H} defined by

$$D(H) = D(-\Delta) = H^2(\mathbb{R}^3), \quad H = -\Delta + V_0.$$

The Kato-Rellich theorem implies that H is self-adjoint on $D(H)$ (for the detail, see Theorem X.15 in [13]). Therefore, we see that $e^{-itH} : \mathcal{H} \rightarrow \mathcal{H}$ is a unitary operator. That is, we have

$$\|e^{-itH}\phi\| = \|\phi\| \quad (1.5)$$

for any $\phi \in \mathcal{H}$. Our first result is concerned with the direct scattering problem for (NLS).

Theorem 1.1. *Assume that*

$$|Q_0| < \mu_0. \quad (1.6)$$

Let $Y_1 = L^3(\mathbb{R}; L^{18/7}(\mathbb{R}^3))$ and $Z_1 = C(\mathbb{R}; \mathcal{H}) \cap Y_1$. Then there exists some $\delta > 0$ such that if $\phi_- \in B(\delta; \mathcal{H})$, then there uniquely exists $u \in Z_1$ such that u is a time-global solution to (1.1) and satisfies

$$u(t) = e^{-itH}\phi_- + \frac{1}{i} \int_{-\infty}^t e^{-i(t-\tau)H} F(u(\tau)) d\tau, \quad (1.7)$$

$$\lim_{t \rightarrow -\infty} \|u(t) - e^{-it\Delta}\phi_-\| = 0. \quad (1.8)$$

Furthermore, there exists a unique $\phi_+ \in \mathcal{H}$ such that

$$\lim_{t \rightarrow \infty} \|u(t) - e^{-it\Delta} \phi_+\| = 0. \quad (1.9)$$

Therefore, the scattering operator for (NLS)

$$S_1 : B(\delta; \mathcal{H}) \ni \phi_- \mapsto \phi_+ \in \mathcal{H}$$

is well-defined.

It is well-known that the wave operators

$$\Omega_{\pm} = s - \lim_{t \rightarrow \pm\infty} e^{itH} e^{it\Delta} : \mathcal{H} \rightarrow P_{ac}(\mathcal{H})$$

and the inverse wave operators

$$\Omega_{\pm}^* = s - \lim_{t \rightarrow \pm\infty} e^{-it\Delta} e^{-itH} P_{ac} : \mathcal{H} \rightarrow \mathcal{H}$$

are well-defined (see Theorem XI.30 in [14]). Here, P_{ac} means the projection onto the absolutely continuous subspace of H . Under condition (1.6), then P_{ac} becomes identity (see Section A below and the proof of Theorem XIII.21,(a) in [15]). We define a mapping S_{V_0} by

$$S_{V_0} = \Omega_+^* \Omega_- : \mathcal{H} \rightarrow \mathcal{H}.$$

The operator S_{V_0} is the scattering operator for (1.1) with $J = -\Delta$ and $f(u) = -V_0 u$.

Using the method of [21, 25, 27, 28, 29], we see that S_{V_0} can be determined from the knowledge of S_1 .

Theorem 1.2. ([21, 25, 27, 28, 29]) *Assume that (1.6) holds. For any $\phi \in \mathcal{H} \setminus \{0\}$, we have*

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} S_1(\varepsilon \phi) = S_{V_0}(\phi) \quad \text{in } \mathcal{H}. \quad (1.10)$$

Once we have determined S_{V_0} , we can reconstruct V_0 , $e^{\pm itH}$, Ω_{\pm} , Ω_{\pm}^* by Enss–Weder [5]. The remaining unknown numbers Q_1 and μ_1 are determined by the following result:

Theorem 1.3. *Assume that (1.6) holds and that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1} \phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.*

(i) *We have*

$$\frac{Q_1}{\mu_1^2} = \frac{\lim_{\lambda \rightarrow \infty} i\lambda^4 \langle (\Omega_+ S_1 \Omega_-^* - id)(\lambda^{-3} \phi_\lambda), \phi_\lambda \rangle}{4\pi \|e^{it\Delta} \phi\|_{(4,4)}^4}. \quad (1.11)$$

(ii) Suppose that $Q_1 \neq 0$. Put

$$\begin{aligned}
 b &= \left| \frac{Q_1}{\mu_1^2} \right|^{1/2}, \quad H(b) = -\Delta - bQ_0 \frac{\exp(-b\mu_0 r)}{r}, \\
 \Psi_1(\alpha) &= \int_{\mathbb{R}} \left\langle \frac{\alpha \exp(-\sqrt{|\alpha|}r)}{r} * |e^{-itH(b)}\phi|^2, |e^{-itH(b)}\phi|^2 \right\rangle dt, \quad \alpha \in \mathbb{R}, \\
 a &= \lim_{\varepsilon \rightarrow 0} i\varepsilon^{-3} b^{-7} \langle (\Omega_+ S_1 \Omega_-^* - id)(\varepsilon\phi_b), \phi_b \rangle, \\
 m_0 &= \max \{m \in \mathbb{Z}_{\geq 0}; \Psi_1(m) \leq |a|\}, \\
 q_1 &= \max \left\{ q = 0, 1; \Psi_1 \left(m_0 + \frac{q}{2} \right) \leq |a| \right\}, \\
 q_{j+1} &= \max \left\{ q = 0, 1; \Psi_1 \left(m_0 + \sum_{k=1}^j \frac{q_k}{2^k} + \frac{q}{2^{j+1}} \right) \leq |a| \right\}, \quad j = 1, 2, \dots
 \end{aligned}$$

Then we have

$$Q_1 = \text{sign} \left(\frac{Q_1}{\mu_1^2} \right) \left(m_0 + \sum_{j=1}^{\infty} \frac{q_j}{2^j} \right). \quad (1.12)$$

Remark 1. We suppose that $V_0 \equiv 0$. Following the proof of Proposition 6 in [18], we can easily show another formula for determining Q_1/μ_1^2

$$\frac{Q_1}{\mu_1^2} = \frac{\lim_{\lambda \rightarrow \infty} \lambda^{-5} \lim_{\varepsilon \rightarrow 0} \frac{i}{\varepsilon^3} \langle (S_1 - id)(\varepsilon\phi_\lambda), \phi_\lambda \rangle}{4\pi \|e^{it\Delta}\phi\|_{(4,4)}^4}. \quad (1.13)$$

Remark that the formula (1.11) is simpler than (1.13).

The contents of this paper is as follows: In Section 2, we show Theorem 1.1. For this purpose, we introduce the L^p - L^q estimate for solutions to the linear Schrödinger equation

$$i\partial_t u + \Delta u + Q_0 \frac{\exp(-\mu_0 r)}{r} u = 0 \quad (1.14)$$

given by Rodnianski–Schlag [16]. From the L^p - L^q estimate, we show that we can treat (NLS) as the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u - \left(Q_1 \frac{\exp(-\mu_1 r)}{r} * |u|^2 \right) u = 0 \quad (1.15)$$

wherever we consider only the direct scattering problem.

In Section 3, we consider some properties of the Schrödinger propagator $e^{-itH(\lambda)}$. Here, $H(\lambda)$ is a self-adjoint operator on \mathcal{H} defined by

$$H(\lambda) = -\Delta + \lambda^2(V_0)_{\lambda^{-1}}.$$

We prove that $e^{-itH(\lambda)}$ satisfies the following properties:

- $\lim_{\lambda \rightarrow \infty} e^{-itH(\lambda)}\phi = e^{it\Delta}\phi$ for some ϕ .
- $\|e^{-itH(\lambda)}\|_6$ is bounded with respect to $\lambda > 0$.

These properties will be used to show Theorem 1.3 in Section 4.

In Section 5, we apply Theorem 1.3 to the inverse scattering problem for the semi-relativistic Hartree equation

$$\left(i\partial_t + \sqrt{1 - \Delta}\right)w + \left(Q_2 \frac{\exp(-\mu_2 r)}{r} * |w|^2\right)w = 0, \quad (t, x) \in \mathbb{R}^{1+3}. \quad (1.16)$$

The existence of the scattering operator can be shown by applying the endpoint Strichartz estimate for the Klein-Gordon equation by Machihara–Nakanishi–Ozawa [9]. We prove that μ_2 and Q_2 can be determined via the formulas (5.6) and (5.7) below. The base of the proof is the following limit:

$$\lim_{\lambda \rightarrow \infty} \left\| e^{it\lambda^2 - it\lambda\sqrt{\lambda^2 - \Delta}}\phi - e^{i\frac{1}{2}\Delta}\phi \right\|_{(4,4)} = 0. \quad (1.17)$$

The functions $e^{-it\lambda\sqrt{\lambda^2 - \Delta}}\phi$ and $e^{i\frac{1}{2}\Delta}\phi$ are solutions to the free semi-relativistic equation and the free Schrödinger equation, respectively. Thus, the limit (1.17) is one of the non-relativistic limit.

2. DIRECT PROBLEM

In this section, we first prepare the Key properties to show Theorems 1.1 and 1.3.

For a measurable function $V : \mathbb{R}^3 \rightarrow \mathbb{C}$, we set

$$\begin{aligned} \|V\|_R &= \sqrt{\int_{\mathbb{R}^{3+3}} \frac{|V(x)V(y)|}{|x-y|^2} d(x, y)}, \\ \|V\|_{\mathcal{K}} &= \sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|V(y)|}{|x-y|} dy. \end{aligned}$$

The norm $\|\cdot\|_R$ is called the Rollnik norm. Remark that the condition (1.6) is equivalent to

$$\frac{|Q_0|}{\mu_0} < 4\pi \min \left\{ \left\| \frac{e^{-r}}{r} \right\|_R^{-1}, \left\| \frac{e^{-r}}{r} \right\|_{\mathcal{K}}^{-1} \right\}. \quad (2.1)$$

For the detail, see Section A below.

Under some suitable condition of V , we obtain the following the time-decay estimate of $e^{-it(-\Delta+V)}$:

Proposition 2.1. ([16]) *Suppose that a measurable function $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfies*

$$\max \left\{ \|V\|_R, \|V\|_{\mathcal{K}} \right\} < 4\pi. \quad (2.2)$$

Then we have

$$\|e^{-it(-\Delta+V)}\phi\|_\infty \leq C|t|^{-\frac{3}{2}}\|\phi\|_1 \quad (2.3)$$

for all $\phi \in L^1(\mathbb{R}^3)$ and $t \neq 0$.

Assume that V satisfies (2.2). By (1.5) and (2.3), it follows from the Riesz–Thorin interpolation theorem that we obtain the L^p – L^q estimate

$$\|e^{-it(-\Delta+V)}\phi\|_p \leq C|t|^{-\frac{3}{2}}\|\phi\|_q \quad (2.4)$$

for all $1 \leq q \leq 2 \leq p \leq \infty$ with $1/p + 1/p = 1$, $\phi \in L^q(\mathbb{R}^3)$ and $t \neq 0$. It is shown by Ginibre–Velo [6] that via T^*T argument, (2.4) gives rise to the class of the Strichartz type estimates.

Proposition 2.2. *Assume that V satisfies (2.2). Let $q_j > 2$, $2 < \beta_j < 6$, $j = 1, 2, 3$. If $2/q_j = 3/2 - 3/\beta_j$, $j = 1, 2, 3$, then we have*

$$\|e^{\pm it(-\Delta+V)}\phi\|_{(q_1, \beta_1)} \leq C\|\phi\|, \quad (2.5)$$

$$\left\| \int_{-\infty}^t e^{\pm i(t-\tau)(-\Delta+V)} f(\tau) d\tau \right\|_{(q_2, \beta_2)} \leq C\|f\|_{(q'_3, \beta'_3)}. \quad (2.6)$$

Here, q'_3 and β'_3 denote the Hölder conjugate of q_3 and β_3 , respectively.

We are ready to show Theorem 1.1.

Proof of Theorem 1.1. Put

$$F(u) = \left(Q_1 \frac{\exp(-\mu_1 r)}{r} * |u|^2 \right) u$$

and

$$\Psi[u](t) = e^{-itH}\phi_- + \frac{1}{i} \int_{-\infty}^t e^{-i(t-\tau)H} F(u(\tau)) d\tau.$$

By (2.1), we have

$$\max\left\{\|V_0\|_R, \|V_0\|_{\mathcal{K}}\right\} < 4\pi.$$

Therefore, from (2.5) and (2.6), we obtain

$$\|\Psi[u]\|_{Z_1} \leq C\left(\|\phi_-\| + \|F(u)\|_{(1,2)}\right).$$

Since $V_0 \in L^{3/2}(\mathbb{R}^3)$, we see that

$$\|F(u)(t)\| \leq \|V_0 * |u(t)|^2\| \|u(t)\|_{18/7} \leq \|V_0\|_{3/2} \|u(t)\|_{18/7}^3,$$

where we have used the Hölder–Young inequality in the second inequality. Hence we see that

$$\|\Psi[u]\|_{Z_1} \leq C\left(\|\phi_-\| + \|u\|_{Z_1}^3\right).$$

Similarly, we obtain

$$\|\Psi[u] - \Psi[\tilde{u}]\|_{Z_1} \leq C\|u - \tilde{u}\|_{Z_1} \left(\|u\|_{Z_1} + \|\tilde{u}\|_{Z_1} \right)^2.$$

It is clear that $\Psi[u] \in C(\mathbb{R}; \mathcal{H})$. Therefore, we see that there uniquely exists $u \in Z_1$ such that $\Psi[u] = u$ for sufficiently small $\delta > 0$. We can immediately find that the fixed point u solves the equation (NLS). Furthermore, we obtain

$$\|u\|_{Z_1} \leq C\|\phi_-\|, \quad (2.7)$$

$$\|u - e^{-itH}\phi_-\|_{Z_1} \leq C\|\phi_-\|^3. \quad (2.8)$$

It follows from $u \in L^3(\mathbb{R}; L^{18/7}(\mathbb{R}^n))$ that

$$\begin{aligned} \|u(t) - e^{-itH}\phi_-\| &\leq \int_t^\infty \|F(u)(t)\| dt \\ &\leq \int_{-\infty}^t \|V_0\|_{3/2} \|u(t)\|_{18/7}^3 dt \\ &\rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Furthermore, if we put

$$S_F(\phi_-) = \phi_- + \frac{1}{i} \int_{\mathbb{R}} e^{itH} (F(u(t))) dt, \quad (2.9)$$

we have

$$\lim_{t \rightarrow +\infty} \|u(t) - e^{-itH} S_F(\phi_-)\| = 0.$$

From $\|\Omega_-(\phi_-)\| = \|\phi_-\|$, there exists some $\delta_0 > 0$ such that if $\phi_- \in B(\delta_0, \mathcal{H})$, then there uniquely exists $v \in Z_1$ satisfying $\Psi[v] = v$ and

$$\lim_{t \rightarrow -\infty} \|v(t) - e^{-itH} \Omega_-(\phi_-)\| = 0. \quad (2.10)$$

Moreover, we have

$$\lim_{t \rightarrow \infty} \|v(t) - e^{-itH} S_F \Omega_-(\phi_-)\| = 0. \quad (2.11)$$

By (1.5), we see from (2.10) and (2.11) that

$$\begin{aligned} &\|v(t) - e^{it\Delta}\phi_-\| \\ &\leq \|v(t) - e^{-itH} \Omega_-(\phi_-)\| + \|e^{-itH} \Omega_-(\phi_-) - e^{it\Delta}\phi_-\| \\ &= \|v(t) - e^{-itH} \Omega_-(\phi_-)\| + \|\Omega_-(\phi_-) - e^{itH} e^{it\Delta}\phi_-\| \\ &\rightarrow 0 \quad \text{as } t \rightarrow -\infty \end{aligned}$$

and

$$\begin{aligned}
& \|v(t) - e^{it\Delta}\Omega_+^* S_F \Omega_-(\phi_-)\| \\
& \leq \|v(t) - e^{-itH} S_F \Omega_-(\phi_-)\| + \|e^{-itH} S_F \Omega_-(\phi_-) - e^{it\Delta}\Omega_+^* S_F \Omega_-(\phi_-)\| \\
& = \|v(t) - e^{-itH} S_F \Omega_-(\phi_-)\| + \|S_F \Omega_-(\phi_-) - e^{itH} e^{it\Delta}\Omega_+^* S_F \Omega_-(\phi_-)\| \\
& \rightarrow 0 \quad \text{as } t \rightarrow +\infty,
\end{aligned}$$

respectively. Thus, (1.8) and (1.9) hold if we define $S_1 = \Omega_+^* S_F \Omega_-$. This completes the proof. \square

3. SCHRÖDINGER PROPAGATOR

For $\lambda > 0$ and $y \in \mathbb{R}^3$, let $H(\lambda, y)$ be a linear operator on \mathcal{H} defined by

$$D(H(\lambda, y)) = D(-\Delta), \quad H(\lambda, y) = -\Delta + \lambda^2 V_0(\lambda x - y).$$

Furthermore, we put

$$H(\lambda) = H(\lambda, 0).$$

The operator $H(\lambda, y)$ becomes a self-adjoint operator on \mathcal{H} . In this section, we list some properties of the Schrödinger propagator $e^{-itH(\lambda, y)}$. The properties are useful to prove Theorem 1.3.

Proposition 3.1. *Let $\lambda > 0$ and $\phi \in \mathcal{H}$.*

(i)

$$(e^{-itH} \phi_\lambda)(x) = \left(e^{-i\lambda^{-2}tH(\lambda)} \phi \right)(\lambda^{-1}x). \quad (3.1)$$

(ii)

$$(e^{-itH(\lambda)} \phi)(x - \lambda^{-1}y) = (e^{-itH(\lambda, y)} \tau_{\lambda^{-1}y} \phi)(x), \quad (3.2)$$

where $\tau_z \phi(x) := \phi(x - z)$, $z \in \mathbb{R}^3$.

Proof. Put $u(t, x) = e^{-itH(\lambda)} \phi(x)$. Then we see that

$$(i\partial_t + \Delta - V)u(\lambda^{-2}t, \lambda^{-1}x) = \lambda^{-2} (i\partial_t u + \Delta u - \lambda^2 V(\lambda \cdot)u)(\lambda^{-2}t, \lambda^{-1}x) = 0$$

and

$$u(0, \lambda^{-1}x) = \phi_\lambda(x).$$

Therefore, we obtain

$$e^{-itH} \phi_\lambda(x) = u(\lambda^{-2}t, \lambda^{-1}x) = e^{-it\lambda^{-2}H(\lambda)} \phi(\lambda^{-1}x).$$

Furthermore, we have

$$\begin{aligned}
& (i\partial_t + \Delta - \lambda^2 V(\lambda(x - \lambda^{-1}y))) u(t, x - \lambda^{-1}y) \\
& = (i\partial_t u + \Delta u - \lambda^2 V(\lambda \cdot)u)(t, x - \lambda^{-1}y) = 0
\end{aligned}$$

and

$$u(0, x - \lambda^{-1}y) = \tau_{\lambda^{-1}y}\phi(x).$$

Therefore, we obtain

$$e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi(x) = u(t, x - \lambda^{-1}y) = e^{-itH(\lambda)}\phi(x - \lambda^{-1}y).$$

□

Proposition 3.2. *If $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, then we have*

$$\lim_{\lambda \rightarrow \infty} \left\| (e^{-itH(\lambda,y)} - e^{it\Delta}) ((-\Delta + i)(-\Delta - i)\phi) \right\| = 0. \quad (3.3)$$

Proof. The proof is essentially similar to that of Theorem VIII.20 in [12]. However, for the sake of completeness, we give here the proof dividing two steps.

(Step I.) For $\alpha \in \mathbb{R}$ and $m \in \mathbb{Z}_{\geq 0}$, let $f(\alpha) = e^{-it\alpha}$ and $g_m(\alpha) = e^{-\alpha^2/m^2}$. Let $\psi \in \mathcal{H}$ and $\varepsilon > 0$. Then there exists some $m_0 \in \mathbb{Z}_{\geq 0}$ such that

$$\|g_m(-\Delta)\psi - \psi\| \leq \varepsilon \quad (3.4)$$

for any $m \geq m_0$. Henceforth, we assume that $m = m_0$. Since g_m is a continuous function vanishing at infinity, it follows from the Stone-Weierstrass theorem (see, e.g., [12]) that there exists some two-parameter polynomial $P(\alpha, \beta)$ such that

$$\sup_{\alpha \in \mathbb{R}} |g_m(\alpha) - P((\alpha + i)^{-1}, (\alpha - i)^{-1})| \leq \varepsilon.$$

Therefore, for any self-adjoint operator A , we have

$$\|g_m(A) - P((A + i)^{-1}, (A - i)^{-1})\| \leq \varepsilon. \quad (3.5)$$

Thus, we obtain

$$\begin{aligned} & \| (g_m(H(\lambda, y)) - g_m(-\Delta))\psi \| \\ & \leq \| g_m(H(\lambda, y)) - P((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1}) \| \|\psi\| \\ & \quad + \| g_m(-\Delta) - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1}) \| \|\psi\| \\ & \quad + \| P((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1})\psi - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1})\psi \| \\ & \leq 2\varepsilon\|\psi\| + \| P((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1})\psi - P((-\Delta + i)^{-1}, (-\Delta - i)^{-1})\psi \|. \end{aligned} \quad (3.6)$$

Here, we have used the property (3.5) in the last inequality. Since it follows that

$$P(\alpha_1, \beta_1) - P(\alpha_2, \beta_2) = \sum_{k,l} C_{k,l} \left\{ (\alpha_1 - \alpha_2) \tilde{P}^{k-1}(\alpha_1, \alpha_2) \beta_1^l + (\beta_1 - \beta_2) \tilde{P}^{l-1}(\beta_1, \beta_2) \alpha_2^k \right\}$$

for some two-parameter polynomial \tilde{P}^k , $k = -1, 0, 1, 2, \dots$, we obtain

$$\begin{aligned} & \|P((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1})\psi - P((- \Delta + i)^{-1}, (- \Delta - i)^{-1})\psi\| \\ & \leq C \{ \|((H(\lambda, y) + i)^{-1} - (- \Delta + i)^{-1})\psi\| + \|((H(\lambda, y) - i)^{-1} - (- \Delta - i)^{-1})\psi\| \}. \end{aligned} \quad (3.7)$$

Henceforth, we put $\psi = (-\Delta + i)(-\Delta - i)\phi$, $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Then we see that

$$\begin{aligned} & \|((H(\lambda, y) \pm i)^{-1} - (-\Delta \pm i)^{-1})\psi\| \\ & \leq \|((H(\lambda, y) \pm i)^{-1} - (-\Delta \pm i)^{-1})(-\Delta + i)(-\Delta - i)\phi\| \\ & = \|(H(\lambda, y) \pm i)^{-1}\lambda^2 V(\cdot - \lambda^{-1}y)(-\Delta \mp i)\phi\| \\ & \leq \|\lambda^2 V(\cdot - \lambda^{-1}y)(-\Delta \mp i)\phi\|. \end{aligned}$$

We set $\eta = \text{dist}(\text{supp}\phi, 0)$. If $\lambda > 0$ is sufficiently large, then we have

$$\|((H(\lambda, y) \pm i)^{-1} - (-\Delta \pm i)^{-1})\psi\| \leq C|Q_0|\lambda^2 \frac{\exp(-\mu_0|\lambda\eta - y|)}{|\lambda\eta - y|} \|(-\Delta \mp i)\phi\|. \quad (3.8)$$

We see from (3.6)–(3.8) that

$$\lim_{\lambda \rightarrow \infty} \|g_m(H(\lambda, y))\psi - g_m(-\Delta)\psi\| = 0. \quad (3.9)$$

(Step II.) Since $f g_m$ is a continuous function vanishing at infinity, it follows from the same argument of the proof of (3.6) that

$$\begin{aligned} & \|(f(H(\lambda, y)) - f(-\Delta))\psi\| \\ & \leq \|(f g_m)(H(\lambda, y))\psi - f(H(\lambda, y))\psi\| + \|(f g_m)(-\Delta)\psi - f(-\Delta)\psi\| \\ & \quad + \|(f g_m)(H(\lambda, y))\psi - (f g_m)(-\Delta)\psi\| \\ & \leq \|f(H(\lambda, y))\| \|g_m(H(\lambda, y))\psi - \psi\| + \|f(-\Delta)\| \|g_m(-\Delta)\psi - \psi\| + 2\varepsilon \|\psi\| \\ & \quad + \left\| \dot{P}((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1})\psi - \dot{P}((- \Delta + i)^{-1}, (- \Delta - i)^{-1})\psi \right\| \\ & \leq \|g_m(H(\lambda, y))\psi - g_m(-\Delta)\psi\| + 2\|g_m(-\Delta)\psi - \psi\| + 2\varepsilon \|\psi\| \\ & \quad + \left\| \dot{P}((H(\lambda, y) + i)^{-1}, (H(\lambda, y) - i)^{-1})\psi - \dot{P}((- \Delta + i)^{-1}, (- \Delta - i)^{-1})\psi \right\| \end{aligned} \quad (3.10)$$

for some two-parameter polynomial \dot{P} . Here, we have used the unitarity of $f(A)$ in the last inequality. By (3.4) and (3.7)–(3.9), (3.3) holds. \square

Proposition 3.3. *For any $\lambda > 0$ and for any $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$, we have*

$$\|e^{-itH(\lambda)}\phi\|_6 \leq C(\phi)\langle t \rangle^{-1}. \quad (3.11)$$

Here, the constant $C(\phi)$ is independent of λ and t .

Proof. Let C_b be the best constant for the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$:

$$C_b = \sup_{\psi \in \dot{H}^1 \setminus \{0\}} \frac{\|\psi\|_6}{\|\nabla \psi\|}.$$

Then we obtain

$$\begin{aligned} \|e^{-itH(\lambda)}\phi\|_6^2 &\leq C_b^2 \|\nabla e^{-itH(\lambda)}\phi\|^2 \\ &\leq C_b^2 \langle \nabla e^{-itH(\lambda)}\phi, \nabla e^{-itH(\lambda)}\phi \rangle \\ &\leq C_b^2 \langle (-\Delta) e^{-itH(\lambda)}\phi, e^{-itH(\lambda)}\phi \rangle \\ &\leq C_b^2 \langle H(\lambda) e^{-itH(\lambda)}\phi, e^{-itH(\lambda)}\phi \rangle - C_b^2 \langle \lambda^2(V_0)_{\lambda^{-1}} e^{-itH(\lambda)}\phi, e^{-itH(\lambda)}\phi \rangle \\ &\leq C_b^2 \langle H(\lambda)\phi, \phi \rangle + C_b^2 \|\lambda^2(V_0)_{\lambda^{-1}}\|_{3/2} \|e^{-itH(\lambda)}\phi\|_6^2 \\ &\leq C_b^2 \|\nabla \phi\|^2 + C_b^2 |\langle \lambda^2(V_0)_{\lambda^{-1}}\phi, \phi \rangle| + C_b^2 \|V_0\|_{3/2} \|e^{-itH(\lambda)}\phi\|_6^2. \end{aligned}$$

It follows that

$$|\langle \lambda^2(V_0)_{\lambda^{-1}}\phi, \phi \rangle| \leq |Q_0| \lambda \frac{\exp(-\mu_0 \eta \lambda)}{\eta} \|\phi\|^2 \leq \frac{|Q_0| \|\phi\|^2}{e \mu_0 \eta^2},$$

where $\eta = \text{dist}(\text{supp } \phi, 0)$. Since

$$C_b^2 \left\| \frac{e^{-r}}{r} \right\|_{3/2} < 1 \quad (3.12)$$

(for the proof, see Section A below), we see that

$$\|e^{-itH(\lambda)}\phi\|_6 \leq \sqrt{\frac{\|\nabla \phi\|^2 + \frac{|Q_0|}{e \mu_0 \eta^2} \|\phi\|^2}{C_b^{-2} - \frac{|Q_0|}{\mu_0} \left\| \frac{e^{-r}}{r} \right\|_{3/2}}}. \quad (3.13)$$

Furthermore, by (2.4) and (3.1), we have

$$\begin{aligned} \|e^{-itH(\lambda)}\phi\|_6 &= \left\| \left(e^{-it\lambda^2 H(\lambda)} \phi \right)_{\lambda^{-1}} \right\|_6 \\ &= \lambda^{-1/2} \left\| e^{-it\lambda^2 H(\lambda)} \phi \right\|_6 \\ &\leq C \lambda^{-1/2} |t\lambda^2|^{-3(1/2-1/6)} \|\phi_\lambda\|_{6/5} \\ &= C \lambda^{-1/2-2+5/2} t^{-1} \|\phi\|_{6/5} \\ &= C t^{-1} \|\phi\|_{6/5}. \end{aligned} \quad (3.14)$$

From (3.13) and (3.14), we have (3.11). \square

4. PROOF OF THEOREM 1.3

As we mention in Section 1, we can reconstruct V_0 from the knowledge of scattering states $(\phi_-, S_1(\phi_-))$. In this section, we give the proof of Theorem 1.3, which enables us to see the exact form of V_1 .

Set

$$K[\phi] = \lim_{\varepsilon \rightarrow 0} \frac{i}{\varepsilon^3} \left\langle (\Omega_+ S_1 \Omega_-^* - id)(\varepsilon \phi), \phi \right\rangle.$$

Using (2.7) and (2.8), we have the following property:

Proposition 4.1. (*Strauss [19]*) *Assume that (1.6) holds. Then we have for all $\phi \in \mathcal{H}$,*

$$K[\phi] = \int_{\mathbb{R}} \left\langle F(e^{-itH} \phi), \phi \right\rangle dt. \quad (4.1)$$

We are now ready to state the proof of Theorem 1.3.

Proof of Theorem 1.3. Remark that we obtain For any $\phi \in \mathcal{H} \setminus \{0\}$ and any $\lambda > \|\phi\|^{3/2} \delta^{-1}$, $(\Omega_+ S_1 \Omega_-^* - id)(\lambda^{-3} \phi_\lambda)$ is well-defined because we have

$$\|\lambda^{-3} \phi_\lambda\| \leq \lambda^{-3/2} \|\phi\|.$$

Let u_λ be the time-global solution to (1.7) satisfying $\phi_- = \lambda^{-3} \phi_\lambda$. Put

$$u_\lambda^0 = e^{it\Delta}(\lambda^{-3} \phi_\lambda), \quad \widetilde{u}_\lambda^0 = e^{it\Delta}(\phi_\lambda), \quad u_\lambda^1 = u_\lambda - u_\lambda^0.$$

Then we obtain

$$i\lambda^4 \left\langle (\Omega_+ S_1 \Omega_-^* - id)(\lambda^{-3} \phi_\lambda), \phi_\lambda \right\rangle = (I)_\lambda + (II)_\lambda^1 + (II)_\lambda^2 + (II)_\lambda^3,$$

where

$$\begin{aligned} (I)_\lambda &= \lambda^4 \int_{\mathbb{R}} \left\langle (V_1 * u_\lambda^0 \overline{u_\lambda^0}) u_\lambda^0, \widetilde{u_\lambda^0} \right\rangle dt, \\ (II)_\lambda^1 &= \lambda^4 \int_{\mathbb{R}} \left\langle (V_1 * u_\lambda^1 \overline{u_\lambda^0}) u_\lambda^0, \widetilde{u_\lambda^0} \right\rangle dt, \\ (II)_\lambda^2 &= \lambda^4 \int_{\mathbb{R}} \left\langle (V_1 * u_\lambda \overline{u_\lambda^1}) u_\lambda^0, \widetilde{u_\lambda^0} \right\rangle dt, \\ (II)_\lambda^3 &= \lambda^4 \int_{\mathbb{R}} \left\langle (V_1 * u_\lambda \overline{u_\lambda}) u_\lambda^1, \widetilde{u_\lambda^0} \right\rangle dt. \end{aligned}$$

Following the proof of Proposition 4.1, we see that for $j = 1, 2, 3$,

$$|(II)_\lambda^j| \leq C\lambda^{-7/2}\|\phi\|^6 \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty.$$

Following Proposition 3.1, we obtain

$$\begin{aligned} (I)_\lambda &= \lambda^{-5} \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} |e^{-i\lambda^{-2}tH(\lambda)} \phi(\lambda^{-1}x - \lambda^{-1}y)|^2 |e^{-i\lambda^{-2}tH(\lambda)} \phi(\lambda^{-1}x)|^2 d(t, x, y) \\ &= \int_{\mathbb{R}^7} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} |e^{-itH(\lambda, y)} \tau_{\lambda^{-1}y} \phi(x)|^2 |e^{-itH(\lambda)} \phi(x)|^2 d(t, x, y) \\ &= \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} \Phi(\lambda, y) dy, \end{aligned}$$

where

$$\Phi(\lambda, y) = \int_{\mathbb{R}^{1+3}} |e^{-itH(\lambda, y)} \tau_{\lambda^{-1}y} \phi(x)|^2 |e^{-itH(\lambda)} \phi(x)|^2 d(t, x).$$

For the function Φ , we have the following property:

Lemma 4.2. *Assume that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1}\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$.*

(i) *For any $y \in \mathbb{R}^3$, we have*

$$\lim_{\lambda \rightarrow \infty} \Phi(\lambda, y) = \|e^{it\Delta} \phi\|_{(4,4)}^4.$$

(ii) *For any $\lambda > 0$ and $y \in \mathbb{R}^3$, we have*

$$|\Phi(\lambda, y)| \leq C(\phi),$$

where the constant $C(\phi)$ is independent of λ and y .

Proof of Lemma 4.2. It follows from the Hölder inequality that

$$\begin{aligned}
& \left| \Phi(\lambda, y) - \|e^{it\Delta}\phi\|_{(4,4)}^4 \right| \\
& \leq \int_{\mathbb{R}^{1+3}} \left(|e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi(x)|^2 |e^{-itH(\lambda)}\phi(x)|^2 - |e^{it\Delta}\phi(x)|^4 \right) d(t, x) \\
& \leq \int_{\mathbb{R}^{1+3}} |e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi(x) - e^{it\Delta}\phi(x)| |e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi(x)| |e^{-itH(\lambda)}\phi(x)|^2 d(t, x) \\
& \quad + \int_{\mathbb{R}^{1+3}} |e^{it\Delta}\phi(x)| |e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi(x) - e^{it\Delta}\phi(x)| |e^{-itH(\lambda)}\phi(x)|^2 d(t, x) \\
& \quad + \int_{\mathbb{R}^{1+3}} |e^{it\Delta}\phi(x)|^2 |e^{-itH(\lambda)}\phi(x) - e^{it\Delta}\phi(x)| |e^{-itH(\lambda)}\phi(x)| d(t, x) \\
& \quad + \int_{\mathbb{R}^{1+3}} |e^{it\Delta}\phi(x)|^3 |e^{-itH(\lambda)}\phi(x) - e^{it\Delta}\phi(x)| d(t, x) \\
& \leq \int_{\mathbb{R}} \|e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi - e^{it\Delta}\phi\| \|e^{-itH(\lambda)}\phi\|_6^3 dt \\
& \quad + \int_{\mathbb{R}} \|e^{it\Delta}\phi\|_6 \|e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi - e^{it\Delta}\phi\| \|e^{-itH(\lambda)}\phi\|_6^2 dt \\
& \quad + \int_{\mathbb{R}} \|e^{it\Delta}\phi\|_6^2 \|e^{-itH(\lambda)}\phi - e^{it\Delta}\phi\| \|e^{-itH(\lambda)}\phi\|_6 dt \\
& \quad + \int_{\mathbb{R}} \|e^{it\Delta}\phi\|_6^3 \|e^{-itH(\lambda)}\phi - e^{it\Delta}\phi\| dt
\end{aligned}$$

where we have used the equality

$$\|e^{-itH(\lambda,y)}\tau_{\lambda^{-1}y}\phi\|_6 = \|e^{-itH(\lambda)}\phi\|_6,$$

which is given by (3.2), in the last inequality. We can easily see that

$$\|e^{it\Delta}\phi\|_6 \leq \langle t \rangle^{-1} (\|\nabla\phi\| + \|\phi\|_{6/5}).$$

Therefore, by Propositions 3.2 and 3.3 and by applying the Lebesgue dominated theorem with respect to the variable t , we obtain (i). Similarly, we have (ii). \square

Let us go back to the proof of Theorem 1.3. Henceforth, we suppose that $\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$ satisfies $\phi \neq 0$ and $(\Delta^2 + 1)^{-1}\phi \in C_c^\infty(\mathbb{R}^3 \setminus \{0\})$. Using the above Lemma 4.2 and Prop A.1,(iii), we see from the Lebesgue dominated theorem with respect to the variable y that

$$\begin{aligned}
\lim_{\lambda \rightarrow \infty} (I)_\lambda &= \|e^{it\Delta}\phi\|_{(4,4)}^4 \int_{\mathbb{R}^3} Q_1 \frac{\exp(-\mu_1|y|)}{|y|} dy \\
&= 4\pi \frac{Q_1}{\mu_1^2} \|e^{it\Delta}\phi\|_{(4,4)}^4,
\end{aligned}$$

which implies (1.11).

We next show (1.12). Suppose that $Q_1 \neq 0$. Recall the definition of b , Ψ , m_0 and q_j , $j = 1, 2, \dots$. It follows from Propositions 4.1 and 3.1, that

$$\begin{aligned}
 a &= b^{-7} K[\phi_b] = b^{-7} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} * |e^{-itH} \phi_b|^2, |e^{-itH} \phi_b|^2 \right\rangle dt \\
 &= b^{-7} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\mu_1 r)}{r} * \left| \left(e^{-itb^{-2}H(b)} \phi \right)_b \right|^2, \left| \left(e^{-itb^{-2}H(b)} \phi \right)_b \right|^2 \right\rangle dt \\
 &= b^{-7+2+3+3} Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-b\mu_1 r)}{br} * |e^{-itH(b)} \phi|^2, |e^{-itH(b)} \phi|^2 \right\rangle dt \\
 &= Q_1 \int_{\mathbb{R}} \left\langle \frac{\exp(-\sqrt{|Q_1|}r)}{r} * |e^{-itH(b)} \phi|^2, |e^{-itH(b)} \phi|^2 \right\rangle dt \\
 &= \Psi_1(Q_1).
 \end{aligned}$$

By the Plancherel theorem, we have

$$\Psi(\alpha) = 4\pi \int_{\mathbb{R}} \int_{\mathbb{R}^3} \frac{\alpha}{|\alpha| + |\xi|^2} \left| \left(\mathfrak{F} |e^{-itH(b)} \phi|^2 \right) (\xi) \right|^2 d\xi dt,$$

where \mathfrak{F} denotes the Fourier transform on \mathcal{H} :

$$\mathfrak{F}\varphi(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} \varphi(x) dx, \quad \varphi \in L^1 \cap \mathcal{H}.$$

Therefore, $\Psi : \mathbb{R} \rightarrow \Psi(\mathbb{R})$ is odd, continuous, bijective and monotonically increasing. Thus, we obtain

$$\Psi \left(m_0 + \sum_{k=1}^j \frac{q_k}{2^k} \right) \leq \Psi(|Q_1|) < \Psi \left(m_0 + \sum_{k=1}^j \frac{q_k}{2^k} + \frac{1}{2^j} \right)$$

and

$$m_0 + \sum_{k=1}^j \frac{q_k}{2^k} \leq |Q_1| < m_0 + \sum_{k=1}^j \frac{q_k}{2^k} + \frac{1}{2^j}.$$

Hence (1.12) holds. \square

5. APPLICATION

In this section, we consider the inverse scattering problem for the semi-relativistic Hartree equation

$$\left(i\partial_t + \sqrt{1 - \Delta} \right) w = F_2(w), \quad (t, x) \in \mathbb{R}^{1+3}. \quad (\text{SRH})$$

Here,

$$F_2(w) = \left(Q_2 \frac{\exp(-\mu_2 r)}{r} * |w|^2 \right) w.$$

The equation (SRH) is used to describe Boson stars. For the detailed physical background, see Lenzmann [7].

There is no result for the inverse scattering problem for the nonlinear semi-relativistic equation. Instead, we review the inverse scattering problem for the nonlinear Klein-Gordon equation. Morawetz–Strauss [10] initially studied the inverse scattering problem for the Klein-Gordon equation with power nonlinearity. Later, Bachelot [2] considered more general cases. Weder [26, 30] proved that a more general class of nonlinearities is uniquely reconstructed, and moreover, a method is given for the unique reconstruction of the potential that acts as a linear operator and that this problem was not considered in [10, 2]. The inverse scattering problem for the Klein-Gordon equation with a cubic convolution

$$\partial_t^2 w - \Delta w + w = (V * |w|^2)w, \quad \text{in } (t, x) \in \mathbb{R}^{1+n}$$

was initially studied by [18]. In the case where V satisfies $V = \lambda|x|^{-\sigma}$ for some σ and λ , [17] proved that V can be recovered. Unfortunately, as far as the author knows, there is no known method to recover the nonlinearity $F_2(w)$.

We shall determine the value of Q_2 and μ_2 from the knowledge of the scattering operator given by the following Proposition:

Proposition 5.1. *Let $s \geq 5/6$. Put $U_2(t) = e^{-it\sqrt{1-\Delta}}$, $X_2 = H^s$, $Y_2 = L^2(\mathbb{R}; H_6^{s-5/6}(\mathbb{R}^3))$ and $Z_2 = C(\mathbb{R}; X_2) \cap Y_2$. Then there exists some $\delta > 0$ satisfying the following properties:*

If $\phi_- \in B(\delta; X_2)$, then there uniquely exist $w \in Z_2$ and $\phi_+ \in X_2$ such that

$$w(t) = U_2(t)\phi_- + \frac{1}{i} \int_{-\infty}^t U_2(t-\tau)F_2(w(\tau))d\tau, \quad (5.1)$$

$$\phi_+ = \phi_- + \frac{1}{i} \int_{\mathbb{R}} U_2(-t)F_2(w(t))dt, \quad (5.2)$$

$$\|w\|_{Z_2} \leq C\|\phi_-\|_{X_2}, \quad (5.3)$$

$$\|w - U_2(t)\phi_-\|_{Z_2} \leq C\|\phi_-\|_{X_2}^3, \quad (5.4)$$

$$\lim_{t \rightarrow \pm\infty} \|w(t) - U_2(t)\phi_{\pm}\|_{X_2} = 0. \quad (5.5)$$

Therefore, we can define the scattering operator for (SRH)

$$S_2 : B(\delta; X_2) \ni \phi_- \mapsto \phi_+ \in X_2.$$

Remark 2. We can easily show Proposition 5.1 by following the proof of Theorem 3.4 in Cho–Ozawa [4].

From the knowledge of $(\phi_-, S_2(\phi_-))$, we give the following formula for determining Q_2/μ_2^2 :

Theorem 5.2. *Let s be a positive number given by Proposition 5.1. Let $1 \leq p < 12/7$ and $k > 11/12$. Assume that*

$$\phi \in (H^s(\mathbb{R}^3) \cap H_p^k(\mathbb{R}^3)) \setminus \{0\}.$$

(i) we have

$$\frac{Q_2}{\mu_2^2} = \frac{\lim_{\lambda \rightarrow \infty} i\lambda^4 \langle (S_2 - id)(\lambda^{-3}\phi_\lambda), \phi_\lambda \rangle}{4\pi \|e^{i\frac{1}{2}\Delta}\phi\|_{(4,4)}^4}. \quad (5.6)$$

(ii) Put

$$\begin{aligned} d &= \left| \frac{Q_2}{\mu_2^2} \right|^{1/2}, \\ \Psi_2(\alpha) &= \int_{\mathbb{R}} \left\langle \frac{\alpha \exp(-\sqrt{|\alpha|r})}{r} * \left| e^{-it\sqrt{d^2-\Delta}}\phi \right|^2, \left| e^{-it\sqrt{d^2-\Delta}}\phi \right|^2 \right\rangle dt, \quad \alpha \in \mathbb{R}, \\ h &= \lim_{\varepsilon \rightarrow 0} i\varepsilon^{-3}d^{-6} \langle (S_2 - id)(\varepsilon\phi_d), \phi_d \rangle, \\ l_0 &= \max \{l \in \mathbb{Z}_{\geq 0}; \Psi_2(l) \leq |h|\}, \\ p_1 &= \max \left\{ p = 0, 1; \Psi_2\left(l_0 + \frac{p}{2}\right) \leq |h| \right\}, \\ p_{j+1} &= \max \left\{ p = 0, 1; \Psi_2\left(l_0 + \sum_{k=1}^j \frac{p_k}{2^k} + \frac{p}{2^{j+1}}\right) \leq |h| \right\}, \quad j = 1, 2, \dots \end{aligned}$$

Then we have

$$Q_2 = \text{sign} \left(\frac{Q_2}{\mu_2^2} \right) \left(l_0 + \sum_{j=1}^{\infty} \frac{p_j}{2^j} \right). \quad (5.7)$$

5.1. Proof of Theorem 5.2. In order to show Theorem 5.2, we first prepare the following lemma:

Lemma 5.3. *For $\lambda > 0$, let*

$$U^\lambda(t) = e^{it\lambda^2 - it\lambda\sqrt{\lambda^2 - \Delta}}, \quad U^\infty(t) = e^{i\frac{1}{2}\Delta}.$$

Assume that $1 \leq p < 12/7$ and $k > 11/12$. If $\phi \in H^s(\mathbb{R}^3) \cap H_p^k(\mathbb{R}^3)$, then we have

$$\lim_{\lambda \rightarrow \infty} \|U^\lambda(t)\phi - U^\infty(t)\phi\|_{(4,4)} = 0. \quad (5.8)$$

Proof. From the embedding $H^{3/4}(\mathbb{R}^3) \hookrightarrow L^4(\mathbb{R}^3)$ and the Plancherel theorem we obtain

$$\begin{aligned} \|U^\lambda(t)\phi - U^\infty(t)\phi\|_4 &\leq C\|U^\lambda(t)\phi - U^\infty(t)\phi\|_{H^{3/4}(\mathbb{R}^3)} \\ &\leq C\left\| \left(e^{it\lambda^2 - it\lambda\sqrt{\lambda^2 + |\xi|^2}} - e^{-it|\xi|^2/2} \right) \langle \xi \rangle^{3/4} \mathcal{F}\phi \right\|_{L^2(\mathbb{R}_\xi^3)}. \end{aligned}$$

Since $\langle \xi \rangle^{3/4} \mathcal{F}\phi \in \mathcal{H}$ and

$$\lambda^2 - \lambda\sqrt{\lambda^2 + |\xi|^2} = \frac{-|\xi|^2}{1 + \sqrt{1 + |\xi|^2/\lambda^2}} \rightarrow \frac{-|\xi|^2}{2} \quad \text{as } \lambda \rightarrow \infty$$

for any $\xi \in \mathbb{R}^3$, it follows from the Lebesgue dominated theorem that

$$\lim_{\lambda \rightarrow \infty} \|U^\lambda(t)\phi - U^\infty(t)\phi\|_4 = 0 \quad \text{for any } t \in \mathbb{R}. \quad (5.9)$$

Now we put $\lambda > 1$. By the $L^p - L^q$ estimate for the free Klein-Gordon equation in [3], we obtain

$$\begin{aligned} \|U^\lambda(t)\phi\|_4 &= \left\| \left(e^{-it\lambda^2\sqrt{1-\Delta}} \phi_\lambda \right)_{\lambda^{-1}} \right\|_4 \\ &= \lambda^{-3/4} \|e^{-it\lambda^2\sqrt{1-\Delta}} \phi_\lambda\|_4 \\ &\leq C\lambda^{-3/4} |t\lambda^2|^{-3/4} \|\phi\|_{H_{4/3}^{5/4}(\mathbb{R}^3)} \\ &\leq C|t|^{-3/4} \|\phi\|_{H_{4/3}^{5/4}(\mathbb{R}^3)}. \end{aligned} \quad (5.10)$$

Using the complex interpolation method for the linear operator $U^\lambda(t)$, we see from

$$\|U^\lambda(t)\phi\|_4 \leq C\|U^\lambda(t)\phi\|_{H^{3/4}(\mathbb{R}^3)} \leq C\|\phi\|_{H^{3/4}(\mathbb{R}^3)}$$

and (5.10) that

$$\|U^\lambda(t)\phi\|_4 \leq C(1 + |t|)^{-3\theta/4} \|\phi\|_{A_\theta}, \quad (5.11)$$

where

$$A_\theta = H^{3/4}(\mathbb{R}^3) \cap H_{p_\theta}^{k_\theta}(\mathbb{R}^3), \quad k_\theta = \frac{3}{4} + \frac{\theta}{2}, \quad p_\theta = \frac{1}{2} + \frac{\theta}{4}, \quad 0 \leq \theta \leq 1.$$

Thus, we can easily see that the left hand side of (5.11) belongs to $L^4(\mathbb{R})$ if $1/3 < \theta \leq 1$. Therefore, we obtain

$$\|U^\lambda(t)\phi - U^\infty(t)\phi\|_4 \leq Cg(t),$$

where $g \in L^4(\mathbb{R})$ is some suitable function independent of λ . By (5.9), it follows from the Lebesgue dominated theorem with respect to time t that (5.8) holds. \square

We are ready to prove Theorem 5.2.

Proof of Theorem 5.2. Following the line of the proof of (1.11), we obtain

$$\begin{aligned} & \lim_{\lambda \rightarrow \infty} i\lambda^4 \left\langle (S_2 - id)(\lambda^{-3}\phi_\lambda), \phi_\lambda \right\rangle \\ &= \lim_{\lambda \rightarrow \infty} i\lambda^{-5} \int_{\mathbb{R}} \left\langle F_2(U_2(t)\phi_\lambda), U_2(t)\phi_\lambda \right\rangle dt \\ &= \lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^{1+2+3}} Q_2 \frac{\exp(-\mu_2|y|)}{|y|} |U^\lambda(t)\phi(x - \lambda^{-1}y)|^2 |U^\lambda(t)\phi(x)|^2 d(t, x, y). \end{aligned}$$

By (5.8), we have (5.6). The remaining formula (5.7) can be shown by the same argument as the proof of (5.6). \square

APPENDIX A. SOME NORMS OF THE YUKAWA POTENTIAL

In this appendix, we consider some norms of the Yukawa potential e^{-r}/r . Our claim is the following:

Proposition A.1. (i) *The condition (1.6) is equivalent to (2.1).*

(ii) *We have (3.12).*

(iii) *We have $\|e^{-r}/r\|_1 = 4\pi$.*

Proof. For $1 \leq p < 3$, we obtain

$$\left\| \frac{e^{-r}}{r} \right\|_p = (4\pi p^{p-3} \Gamma(3-p))^{1/p}.$$

Here, Γ is the usual Gamma function. In particular, we see that

$$\left\| \frac{e^{-r}}{r} \right\|_1 = 4\pi, \quad \left\| \frac{e^{-r}}{r} \right\|_{3/2} = \frac{2^{5/3}\pi}{3}.$$

Hence we have proved (iii). By Aubin and Talenti [1, 20], the best constant C_b is explicitly given by

$$C_b = \frac{1}{3\pi} \left(\frac{4}{\sqrt{\pi}} \right)^{2/3}.$$

Thus, we have

$$C_b^2 \left\| \frac{e^{-r}}{r} \right\|_{3/2} = \frac{8\pi^{-1/3}}{9} < 1,$$

which implies (ii).

We now consider the Rollnik norm of the Yukawa potential. Lieb [8] proved that the best constant for the Hardy-Littlewood-Sobolev inequality is given by

$$\sup \left\{ \frac{\|\psi * r^{-2}\|_3}{\|\psi\|_{3/2}}; \psi \in L^{3/2}(\mathbb{R}^3) \setminus \{0\} \right\} = 2^{2/3} \pi^{4/3}.$$

Therefore, we see from the Hölder inequality that

$$\begin{aligned} \left\| \frac{e^{-r}}{r} \right\|_R &= \sqrt{\left\langle \frac{e^{-r}}{r} * \frac{1}{r^2}, \frac{e^{-r}}{r} \right\rangle} \leq \left\| \frac{e^{-r}}{r} \right\|_{3/2}^{1/2} \left\| \frac{e^{-r}}{r} * \frac{1}{r^2} \right\|_3^{1/2} \\ &\leq 2^{1/3} \pi^{2/3} \left\| \frac{e^{-r}}{r} \right\|_{3/2} = 4\pi \frac{\pi^{2/3}}{3} < 4\pi. \end{aligned} \quad (\text{A.1})$$

On the other hand, we next consider $\|e^{-r}/r\|_{\mathcal{K}}$. Since the function $(e^{-r}/r) * r^{-1}$ is radial, we have

$$\begin{aligned} \frac{e^{-r}}{r} * \frac{1}{r}(x) &= \int_{\mathbb{R}^3} \frac{e^{-|y|}}{|y| \sqrt{||x| - y_1|^2 + |y_2|^2 + |y_3|^2}} dy \\ &= \int_0^\infty \int_{\mathbb{R}S^2} \frac{e^{-R}}{R \sqrt{||x| - \theta_1|^2 + |\theta_2|^2 + |\theta_3|^2}} d\sigma(\theta) dR \\ &= \int_0^\infty R^{-1} e^{-R} \int_{\mathbb{R}S^2} \frac{d\sigma(\theta)}{\sqrt{||x| - \theta_1|^2 + |\theta_2|^2 + |\theta_3|^2}} dR. \\ &= \int_0^\infty R^{-1} e^{-R} \int_{-R}^R \int_{\sqrt{R^2 - s^2} \mathbb{S}^1} \frac{d\sigma(\vartheta)}{\sqrt{||x| - s|^2 + R^2 - s^2}} \frac{R ds}{\sqrt{R^2 - s^2}} dR. \\ &= 4\pi \int_0^\infty e^{-R} \frac{\min\{R, |x|\}}{|x|} dR = \frac{4\pi}{|x|} (1 - e^{-|x|}). \end{aligned}$$

Thus, we obtain

$$\left\| \frac{e^{-r}}{r} \right\|_{\mathcal{K}} = 4\pi. \quad (\text{A.2})$$

Hence (ii) is true. \square

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